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On the necessity and sufficiency of *PLUS* factorizations

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Abstract

PLUS factorizations, or customizable triangular factorizations, of nonsingular matrices have found applications in source coding and computer graphics. However, there are still some open problems. In this paper, we present a new necessary condition and a sufficient condition for the existence of generic *PLUS* factorizations.

Keywords: Triangular factorization; Reversible integer transform; Transform by shears

1. Introduction

PLUS factorizations (or customizable triangular factorizations), introduced by Hao [4], come from the theories of triangular factorizations of a general nonsingular matrix over an abstract algebraic structure [5,10], and their applications in computer graphics and lossless coding, such as the rotation by shears [1,9], perfect reversible

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integer transform [2,3], and so on. *PLUS* factorizations encompass and generalize quite a few triangular factorizations of nonsingular matrices [3,8–10].

A *PLUS* factorization for an arbitrary nonsingular N-by-N matrix $\boldsymbol{A}$ has the form of:

$$\boldsymbol{A} = \boldsymbol{PLUS}$$

where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a unit lower triangular matrix, $\boldsymbol{U}$ is an upper triangular matrix whose diagonal entries are prescribed as long as the determinant is equal to that of $\boldsymbol{A}$ up to a possible sign adjustment, and $\boldsymbol{S}$ is a unit lower triangular matrix of which all but $N-1$ off-diagonal entries are set to zeros and the positions of those $N-1$ entries are also flexibly customizable. For example, $\boldsymbol{S}$ can be a single-row, a single-column, a bidiagonal matrix, or a specially-patterned matrix. A pseudo-permutation matrix [4] can take the role of the permutation matrix $\boldsymbol{P}$ as well: it is a simple unit upper triangular matrix with off-diagonal elements being 0, 1 or -1. Besides *PLUS*, a customizable factorization also has other alternatives—*LUSP*, *PSUL* or *SULP* for lower $\boldsymbol{S}$, and *PULS*, *ULSP*, *PSLU*, *SLUP* for upper $\boldsymbol{S}$, generally still referred to as *PLUS* factorizations.

In summary, *PLUS* factorizations are called customizable in that: (i) all diagonal entries of $\boldsymbol{U}$, say $d_1, \ldots, d_N$, are customizable as long as $\prod_{i=1}^{N} d_i = \det(\boldsymbol{P}^{-1}\boldsymbol{A})$; (ii) various types of factorizations are available for customers' different purposes, such as *PLUS*, *LUSP*, etc.; (iii) most importantly, the lower triangular structure of $\boldsymbol{S}$ is customizable, that is, the positions of all its nonzero off-diagonal entries can be pre-designated to some extent. We focus on (iii) in this paper.

In [4], three necessary and sufficient conditions corresponding to the special cases of single-row, single-column, and bi-diagonal $\boldsymbol{S}$ for *PLUS* factorizations have been given, and a necessary condition regarding the structure of $\boldsymbol{S}$ has also been presented for generic *PLUS* factorizations.

In this paper, we first present a counterexample to demonstrate the insufficiency of the necessary condition in [4], then introduce a new necessary constraint, and derive a sufficient condition for generic *PLUS* factorizations.

To simplify our discussion, we make a few simplifications to the customizability *without any loss of generality*: (i) the number of nonzero off-diagonal elements in lower triangular $\boldsymbol{S}$ is limited to $N-1$; (ii) $\boldsymbol{P}$ is a row permutation matrix; (iii) $d_1 = d_2 = \cdots = d_{N-1} = 1$, i.e., all diagonal entries of $\boldsymbol{U}$ are designated to be 1's except the last one. It is not hard to see that the generalizations with any of these constraints dropped are trivial. Thus, our problem is reduced to whether all the first $N-1$ leading principal minors for any nonsingular matrix $\boldsymbol{A}$ can be customized to 1's simply by row permutations and a series of elementary column operations determined by $\boldsymbol{S}$.

It's worth mentioning that, strictly speaking, when $\boldsymbol{P}$ is a pure permutation matrix, each diagonal entry of $\boldsymbol{U}$ is customized up to a possible sign difference. We acquiesce in this point during the whole customization–factorization process. However, as for our conclusions in this paper, it's easy to know that this kind of sign uncertainty can be restricted to the first or the last diagonal entry only.

2. Necessity

Hao's necessary condition [4] describes how the nonzero elements are patterned in a special matrix $\boldsymbol{S}$ if the generic *PLUS* factorizations exist for $\boldsymbol{S}$. In this section, we give a new characterization of this necessary condition and present another one.

2.1. Another characterization of Hao's necessary condition

Hereinafter, we use $NZ_{\boldsymbol{S}}$, the possibly nonzero off-diagonal entry set of $\boldsymbol{S}$, to denote its customization structure. (Of course, some positions in $NZ_{\boldsymbol{S}}$ may still be zero as a result of computation.) Let $|\,|$ denote the cardinality of a set. Then we have $|NZ_{\boldsymbol{S}}| = N - 1$ from our assumptions. Note that $NZ_{\boldsymbol{S}}$ is not ordered.

If we use $\boldsymbol{S}_{i,j}(x)$ (or $\boldsymbol{S}_{i,j}$) to represent a special lower-triangular matrix with the (i, j)th entry x and all others equal to zero, then $\boldsymbol{S}$ can be expressed as the product of a series of $\boldsymbol{S}_{i,j}$: $\boldsymbol{S} = \prod_{k=1}^{N-1} \boldsymbol{S}_{i_k,j_k}(x_k)$, where x_k are the values at the customized positions $(i_k, j_k) \in NZ_{\boldsymbol{S}}$, $1 \leqslant k \leqslant N - 1$. Obviously, $\boldsymbol{S}_{i,j}^{-1}(x) = \boldsymbol{S}_{i,j}(-x)$. Thus if there exists a *PLUS* factorization $\boldsymbol{A} = \boldsymbol{PLUS}$, then, after a series of elementary column operations (determined by $NZ_{\boldsymbol{S}}$) applied to $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{A}$, the first through $(N - 1)$th leading principal minors of the transformed matrix $\boldsymbol{P}^{T}\boldsymbol{A}\boldsymbol{S}^{-1}$ become 1's. The left $\boldsymbol{P}$ is to assure the existence (solvability) of all needed column operations.

Hao used a customization matrix

$$\boldsymbol{B}: \quad \boldsymbol{B}(i; j) = \begin{cases} 1, & (i, j) \in NZ_{\boldsymbol{S}}, \\ 0, & (i, j) \notin NZ_{\boldsymbol{S}}, \end{cases}$$

to characterize the structure of $\boldsymbol{S}$. It is a triangular Boolean matrix indicating all customization positions of $\boldsymbol{S}$. The necessary condition for generic *PLUS* factorizations given in [4] is formulated as

$$\sum_{j=1}^{k} \sum_{i=j+1}^{N} \boldsymbol{B}(i; j) \geqslant k \quad \text{and} \quad \sum_{i=1}^{k} \sum_{j=1}^{N-i} \boldsymbol{B}(N - i + 1; j) \geqslant k.$$

Though somewhat complicated in form, it seems intuitively obvious. See Fig. 1 for an example of $N = 4$.

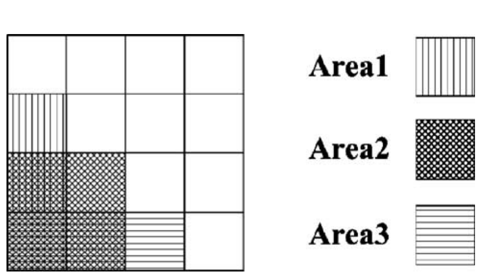

Fig. 1. Illustration of Hao's necessary condition ($N = 4$).

To make the first three leading principal minors of a *general* 4×4 matrix all equal to 1, we need at least three independent unknowns in $\boldsymbol{S}$. More specifically, to customize the third leading principal minor, $NZ_{\boldsymbol{S}}$ must have an element situated in Area3; due to the same reason, there also exists (at least) another element of $NZ_{\boldsymbol{S}}$ in Area2 to customize the second leading principal minor, besides the above one used already; finally, the first leading principal minor should also have its own associated element located in Area1. Summarizing all these requirements and describing them with the customization matrix, we obtain the necessary condition given by Hao. (In fact, a rigorous proof can be made from this intuition.)

2.2. A counterexample and another necessary condition

It is fair to ask if the above necessary condition is sufficient. Unfortunately, this is not true in general.

Let $N = 4$, $NZ_{\boldsymbol{S}} = \{(2, 1), (3, 1), (3, 2)\}$, i.e.,

$$\boldsymbol{S} = \begin{bmatrix} 1 & & & \\ \times & 1 & & \\ 0 & 0 & 1 & \\ \times & \times & 0 & 1 \end{bmatrix}.$$

We shall prove that, given a matrix $\boldsymbol{A}$ satisfying $\boldsymbol{A}(1; 1) = \boldsymbol{A}(1; 2) = \boldsymbol{A}(1; 4) = \boldsymbol{A}(2; 3) = \boldsymbol{A}(3; 3) = \boldsymbol{A}(4; 3) = 0$, $\boldsymbol{A}(1; 3) = a$, $|a| \neq 0$ or 1, i.e.,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & a & 0 \\ * & * & 0 & * \\ * & * & 0 & * \\ * & * & 0 & * \end{bmatrix},$$

the *PLUS* factorization $\boldsymbol{A} = \boldsymbol{PLUS}$ does not exist; or equivalently, there exists no row permutation matrix $\boldsymbol{P}$ such that the first to third leading principal minors of $\boldsymbol{A}' = \boldsymbol{PAS}^{-1}$ are all equal to 1.

As a matter of fact, from the structure of $\boldsymbol{S}$, we easily know that, any element located in the same row or the same column of a keeps unchanged after the transformation. Let $\boldsymbol{A}_k$ denote the kth leading principal submatrix of $\boldsymbol{A}$. Noticing that $\boldsymbol{A}'_1$, $\boldsymbol{A}'_2, \boldsymbol{A}'_3$ are all invertible, we can locate the final position of a in $\boldsymbol{A}'$. In fact, seen from both the row structure and the column structure, a must be $\boldsymbol{A}'(3; 3)$, i.e.,

$$\boldsymbol{A}' = \begin{bmatrix} * & * & 0 & * \\ * & * & 0 & * \\ 0 & 0 & a & 0 \\ * & * & 0 & * \end{bmatrix}.$$

However, now the third leading principal minor of $\boldsymbol{A}'$ is not 1 if $|\boldsymbol{A}'_3| = a|\boldsymbol{A}'_2| = \pm a \neq \pm 1$. As a result, the corresponding *PLUS* factorization does not exist even if $\boldsymbol{S}$ satisfies Hao's necessary condition.

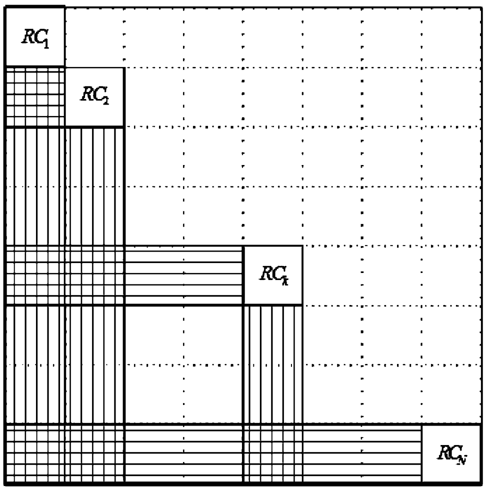

Fig. 2. Illustration of RC necessary condition.

It is quite easy to generalize the above counterexample to an arbitrary N. Let RC_k denote the set of all positions below the diagonal and in the same row or the same column of $\boldsymbol{S}(k;k)$, i.e.,

$$RC_k = \{(i,j) | i = k \text{ or } j = k,\ j < i\}, \quad k = 1, \ldots, N.$$

The example tells us that for any k, there must exist an element in RC_k belonging to $NZ_{\boldsymbol{S}}$ (see Fig. 2). Thus we obtain another necessity theorem as follows:

Theorem 1. *PLUS factorization* $\boldsymbol{A} = \boldsymbol{PLUS}$ *exists only if* $RC_k \cap NZ_{\boldsymbol{S}} \neq \phi$ *for all* $k : 1 \leqslant k \leqslant N$. (*This necessary condition is denoted as* (N1) *below*.)

We can also use RC_k notations to represent Hao's necessary condition (see Fig. 1):

$$\left|\left(\bigcup_{i=1}^{k} RC_i\right) \cap NZ_{\boldsymbol{S}}\right| \geqslant k, \quad \left|\left(\bigcup_{j=N+1-k}^{N} RC_j\right) \cap NZ_{\boldsymbol{S}}\right| \geqslant k,$$
$$1 \leqslant k \leqslant N-1. \tag{N2}$$

(N1) together with (N2) constitutes our new necessary condition for *PLUS* factorizations.

3. Sufficiency

As mentioned above, for generic *PLUS* factorizations, the sufficiency problem is more important and meaningful in practice. That is, under what conditions can we perform such factorizations for an arbitrary nonsingular matrix? From the viewpoint of equation solving we may ask: what kind of structure of $\boldsymbol{S}$ can guarantee that there always exists a row permutation $\boldsymbol{P}$ such that the system of $N-1$ equations $(\boldsymbol{PAS}^{-1})_k = 1$, $1 \leqslant k \leqslant N-1$, has at least one solution given any nonsingular $\boldsymbol{A}$? In the worst case, the degree of this system can be $N-1$.

As a structure characterization of the special matrix $\boldsymbol{S}$, $NZ_{\boldsymbol{S}}$ is not ordered. This more or less increases the difficulty in solving our problem. From the matrix factorization $\boldsymbol{S} = \boldsymbol{S}_{i_1,j_1}\boldsymbol{S}_{i_2,j_2}\cdots\boldsymbol{S}_{i_{N-1},j_{N-1}}$, the only order constraint is that if $j_p = i_q$, then S_{i_q,j_q} should appear left to S_{i_p,j_p} (not necessarily adjacent); other than that, the matrices can be arranged arbitrarily. Note that in most cases there is a definite one-to-one relationship between $NZ_{\boldsymbol{S}}$ and the first $N-1$ leading principal minors: each element of $NZ_{\boldsymbol{S}}$ plays its own role in customizing some leading principal minor. Thus we may find an *ordered sequence* (or *sequence*, for short) of $\boldsymbol{S}$ by arranging all the elements from the one customizing the first leading principal minor to the one customizing the $(N-1)$th leading principal minor. Define the set $Seq_{\boldsymbol{S}}$ by:

$$\begin{aligned} Seq_{\boldsymbol{S}} = \{((i_1, j_1), (i_2, j_2), \ldots, (i_{N-1}, j_{N-1}))|(i_k, j_k) \in NZ_{\boldsymbol{S}},\\ j_k \leqslant k < i_k,\ j_p = i_q \to q < p\}. \end{aligned}$$

Each sequence of $Seq_{\boldsymbol{S}}$ represents a possible operating process by $\boldsymbol{S}$ on $\boldsymbol{A}$. Clearly, $\boldsymbol{S} = \boldsymbol{S}_{i_1,j_1}\boldsymbol{S}_{i_2,j_2}\cdots\boldsymbol{S}_{i_{N-1},j_{N-1}}$ holds for any sequence $sq = ((i_1, j_1), \ldots, (i_{N-1}, j_{N-1})) \in Seq_{\boldsymbol{S}}$. $|Seq_{\boldsymbol{S}}|$ is not necessarily 1. For instance, given

$$\boldsymbol{S} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \times & & 1 & \\ \times & \times & & 1 \end{bmatrix},$$

we have $Seq_{\boldsymbol{S}} = \{((3,1),(4,1),(4,2)), ((3,1),(4,2),(4,1)), ((4,1),(3,1),(4,2))\}$. Note that since $\boldsymbol{S}^{-1} = \boldsymbol{S}^{-1}_{i_{N-1},j_{N-1}}\cdots\boldsymbol{S}^{-1}_{i_1,j_1}$, $\boldsymbol{S}$ acts on $\boldsymbol{A}$ in the reverse order of sq.

Definition (*Valid sequence*). Given $sq = ((i_1, j_1), (i_2, j_2), \ldots, (i_{N-1}, j_{N-1})) \in Seq_{\boldsymbol{S}}$, if

(a) $i_{N-1} = N$; and
(b) for any $k : 1 \leqslant k < N-1$,
  (b.1) $i_k = k+1$; or
  (b.2) $i_k = i_{k+1}$ only if $j_{k+1} = k+1$,

then sq is called a *valid sequence* (VS) of $\boldsymbol{S}$ or $NZ_{\boldsymbol{S}}$.

By definition $\{i_k\}$ is increasing, i.e., $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{N-1}$.

Theorem 2. *Given a structure $NZ_{\boldsymbol{S}}$, if it has a VS $sq = ((i_1, j_1), (i_2, j_2), \ldots, (i_{N-1}, j_{N-1}))$, then the PLUS factorization $\boldsymbol{A} = \boldsymbol{PLUS}$ holds for any nonsingular matrix $\boldsymbol{A}$.*

Proof. Suppose $i_p \neq i_{p+1} = \cdots = i_q \neq i_{q+1}$, then $i_p = p+1$ and $i_q = q+1$. Since $\{i_k\}$ is increasing, the column operations associated with (i_1, j_1) through (i_p, j_p) (or (i_{p+1}, j_{p+1}) through (i_q, j_q)) do not change the kth leading principal minor for any $k \geqslant p+1$ (or $k \geqslant q+1$). Thus it suffices to show that there exist $\boldsymbol{S}_{i_{p+1},j_{p+1}}, \ldots, \boldsymbol{S}_{i_q,j_q}$ and a row permutation $\boldsymbol{P}$ associated with the first $q+1$ rows only such that for any $\boldsymbol{A}'$ with its $(q+1)$th leading principal submatrix nonsingular, the kth $(p+1 \leqslant k < q)$ leading principal minors of $\boldsymbol{P}\boldsymbol{A}'\boldsymbol{S}_{i_q,j_q}^{-1} \cdots \boldsymbol{S}_{i_{p+1},j_{p+1}}^{-1}$ are all equal to 1.

Without loss of generality, we assume $q - p > 1$, then $j_q = q$, $j_{q-1} = q-1, \ldots, j_{p+2} = p+2$. Since

$$\begin{aligned}\{1, 2, \ldots, p+1\} \cup \{i_{p+1}\} \backslash \{j_{p+1}\} &\subset \{1, 2, \ldots, p+2\} \cup \{i_{p+2}\} \backslash \{j_{p+2}\} \\ &\subset \cdots \subset \{1, 2, \ldots, q\} \cup \{i_q\} \backslash \{j_q\} \\ &\subset \{1, 2, \ldots, q+1\}, \qquad (*)\end{aligned}$$

a series of row permutations can be found one by one, from q to $p+1$ to guarantee each $\boldsymbol{S}_{i_k,j_k}$ $(q \geqslant k \geqslant p+1)$ solvable for any $\boldsymbol{A}'$ with its $(q+1)$th leading principal submatrix nonsingular. Then, following the order of sq, all column operations can be properly determined. The proof is now complete. □

Intuitively, the above theorem requires that (i) each row (except the first) of $\boldsymbol{S}$ has one customization position, or (ii) in the case that there is more than one customization position in a single row, all but the left-most one are next to each other at the right-most end. It is not difficult to verify that VS satisfies (N1) and (N2).

We might ask how to find a potential VS in Seq_S. By induction on N, we can easily show that:

Proposition 1. *If there exists a VS in Seq_S, then $|Seq_S| = 1$.*

On the other hand, we can always get a sequence of $\boldsymbol{S}$ by rearranging all entries of NZ_S left-to-right, from the first row to the last. Hence we only need to consider such a special sequence when judging by Theorem 2. Finally, it's worth mentioning that we can also generalize Theorem 2 to block matrices; see [6] or Appendix A.

4. Examples

We demonstrate our necessary and sufficient conditions with an simple example of $N = 4$. According to our sufficient condition and our necessary condition, all the 20 possibilities of the customizable special matrix $\boldsymbol{S}$ for *PLUS* factorizations can be classified into 3 classes, with 12, 4 and 4 cases, respectively.

Class A (12 cases):

$$\underset{\text{(A1)}}{\begin{bmatrix}1&&&\\&1&&\\&&1&\\\times&\times&\times&1\end{bmatrix}}\quad\underset{\text{(A2)}}{\begin{bmatrix}1&&&\\\times&1&&\\&&1&\\&\times&\times&1\end{bmatrix}}\quad\underset{\text{(A3)}}{\begin{bmatrix}1&&&\\\times&1&&\\&&1&\\\times&&\times&1\end{bmatrix}}\quad\underset{\text{(A4)}}{\begin{bmatrix}1&&&\\&1&&\\\times&\times&1&\\&&\times&1\end{bmatrix}}$$

$$\underset{\text{(A5)}}{\begin{bmatrix}1&&&\\\times&1&&\\&\times&1&\\&&\times&1\end{bmatrix}}\quad\underset{\text{(A6)}}{\begin{bmatrix}1&&&\\\times&1&&\\\times&&1&\\&&\times&1\end{bmatrix}}\quad\underset{\text{(A7)}}{\begin{bmatrix}1&&&\\\times&1&&\\&\times&1&\\&\times&&1\end{bmatrix}}\quad\underset{\text{(A8)}}{\begin{bmatrix}1&&&\\&1&&\\\times&\times&1&\\&\times&&1\end{bmatrix}}$$

$$\underset{\text{(A9)}}{\begin{bmatrix}1&&&\\\times&1&&\\\times&&1&\\&\times&&1\end{bmatrix}}\quad\underset{\text{(A10)}}{\begin{bmatrix}1&&&\\&1&&\\\times&\times&1&\\\times&&&1\end{bmatrix}}\quad\underset{\text{(A11)}}{\begin{bmatrix}1&&&\\\times&1&&\\&\times&1&\\\times&&&1\end{bmatrix}}\quad\underset{\text{(A12)}}{\begin{bmatrix}1&&&\\\times&1&&\\\times&&1&\\\times&&&1\end{bmatrix}}.$$

Class B (4 cases):

$$\underset{\text{(B1)}}{\begin{bmatrix}1&&&\\&1&&\\\times&&1&\\\times&&\times&1\end{bmatrix}}\quad\underset{\text{(B2)}}{\begin{bmatrix}1&&&\\&1&&\\&\times&1&\\&\times&\times&1\end{bmatrix}}\quad\underset{\text{(B3)}}{\begin{bmatrix}1&&&\\\times&1&&\\&&1&\\\times&\times&&1\end{bmatrix}}\quad\underset{\text{(B4)}}{\begin{bmatrix}1&&&\\\times&1&&\\\times&\times&1&\\&&&1\end{bmatrix}}.$$

Class C (4 cases):

$$\underset{\text{(C1)}}{\begin{bmatrix}1&&&\\&1&&\\\times&&1&\\&\times&\times&1\end{bmatrix}}\quad\underset{\text{(C2)}}{\begin{bmatrix}1&&&\\&1&&\\&\times&1&\\\times&&\times&1\end{bmatrix}}\quad\underset{\text{(C3)}}{\begin{bmatrix}1&&&\\&1&&\\&\times&1&\\\times&\times&&1\end{bmatrix}}\quad\underset{\text{(C4)}}{\begin{bmatrix}1&&&\\&1&&\\\times&&1&\\\times&\times&&1\end{bmatrix}}.$$

From our conclusions, a *PLUS* factorization is guaranteed for any of the 12 cases in (A), but does not exist for any of the 4 cases in (B). By contrast, the conditions in [4] only certify 3 cases—(A1), (A5), and (A12), and negate two cases—(B2) and (B4).

However, our conditions are not capable of judging any case in group (C). Particularly, for (C4), Seq_S has three elements, but in any situation the last column operation will inevitably affect some leading principal minor customized already.

In the following we show that the *PLUS* factorization does exist for (C3), therefore our sufficient condition obtained so far (Theorem 2) is not yet a necessary and sufficient one. In (C3), $NZ_S = \{(3, 2), (4, 1), (4, 2)\}$, $((4, 1), (3, 2), (4, 2)) \in Seq_S$. We only list the steps in making appropriate row permutations. First, select three rows from $\boldsymbol{A}$ to make $(\boldsymbol{P}_1\boldsymbol{A})(1, 2, 3; 1, 3, 4) \equiv \boldsymbol{B}_1$ invertible; denote $\boldsymbol{P}_1\boldsymbol{A}$ by $\boldsymbol{A}_1$. Second, by Laplace expansion, there must exist a nonzero element in the last row of $\boldsymbol{B}_1$ with its minor (in $\boldsymbol{B}_1$) also nonzero. Thus we can obtain $\boldsymbol{A}_2 = \boldsymbol{P}_2\boldsymbol{A}_1$ with $\boldsymbol{A}_2(1; 4)$ nonzero and $\boldsymbol{A}_2(2, 3; 1, 3)$ nonsingular. This is a guarantee to make $(\boldsymbol{P}_3\boldsymbol{A}_2)(1, 2; 1, 3)$ invertible in the final step by a $\boldsymbol{P}_3$ regarding the second row and the third row only. As a result *PLUS* factorization exists for (C3).

In conclusion, our sufficient condition is not strong enough to deal with all cases. We expect to give a more general sufficient condition, or a necessary and sufficient one, as the future direction of our work.

Appendix A. Generalization of *PLUS* factorizations to block matrices

In Section 4.3 of [6] we also generalized our problem and Theorem 2 to block matrices.

Lemma. *For any $m \times n$ matrix $\boldsymbol{A}$ with full row rank, if $q \leqslant p < m$, then there exists a row permutation such that $(\boldsymbol{PA})(1, 2, \ldots, m-p; 1, 2, \ldots, n-q)$ has full row rank.*

Proof. Since $q \leqslant p$, the rank of the submatrix $\boldsymbol{A}(1, 2, \ldots, m; 1, 2, \ldots, n-q)$ is not less than $m-p$. Then we can pick up $m-p$ rows in this submatrix which are linearly independent. □

By this lemma and the relation $(*)$ in the proof of Theorem 2, we can easily establish a block version of Theorem 2 as a generalization, for any nonsingular block matrix with square diagonal blocks, the size of which is non-decreasing from top to bottom. (Note in this block version of $\boldsymbol{A} = \boldsymbol{PLUS}$, all structures of $\boldsymbol{S}$, $\boldsymbol{L}$, $\boldsymbol{U}$, and $\boldsymbol{A}$ are characterized at the block level, while $\boldsymbol{P}$ is still a permutation matrix at the *element* level.) Interested readers may refer to [6,7] for more details.

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